On the symmetry group of the n-dimensional Berwald-Moór metric

Hengameh Raeisi-Dehkordi* and Mircea Neagu†

Abstract

In this paper we parametrize the symmetry group of the n-dimensional Berwald-Moór metric. Some properties of this Lie group are studied, and its corresponding Lie algebra is computed.

Mathematics Subject Classification (2010): 53C60, 54H15, 17B45.
Key words and phrases: tangent bundle, n-dimensional Berwald-Moór metric, symmetry group, Lie group, Lie algebra.

1 Introduction

The geometrical Berwald-Moór structure ([3], [9]) was intensively investigated by P.K. Rashevski ([13]) and further physically fundamented and developed by G.S. Asanov ([1]), D.G. Pavlov and G.I. Garas'ko ([12], [5]). These physical studies emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions in the theory of space-time structure, gravitation and electromagnetism. In such a context, one underlines the important role played by the Berwald-Moór metric

$$F_n : TM^n \to \mathbb{R}, \qquad F_n(y) = \sqrt[n]{y^1 y^2 ... y^n}, \qquad n \geq 2,$$

whose Finslerian geometry is studied on tangent bundles by M. Matsumoto and H. Shimada ([7]), and, in a jet geometrical approach, by V. Balan and M. Neagu ([2]). It is a well-known fact that, from a physical point of view, an Einstein relativistic law says that the form of all physical laws must be the same in any inertial reference frame (local chart of coordinates). For such a reason, we study in this paper the geometrical coordinate transformations which keep unchanged the Berwald-Moór metric of order $n \geq 3$. The particular two and three dimensional cases are deeply studied in [11]. Notice that the geometrical translation of the previous Einstein's physical law is that any geometrical object used in that physical theory must have the same local form in any local chart of coordinates.

*Department of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Ave., 15875-4413 Tehran, Iran, E-mail: hengameh_62@aut.ac.ir

†Department of Mathematics and Informatics, University Transilvania of Braşov, 50 Iuliu Maniu Blvd., 500091 Braşov, Romania, E-mail: mircea.neagu@unitbv.ro

2 The symmetry group of the Berwald-Moór metric of order n

We remind that $(x, y) = (x^i, y^i)$ are the coordinates of the tangent bundle TM^n (associated to an n-dimensional real manifold M^n), which transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \qquad \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \tag{1}$$

where $i, j = \overline{1, n}$ and rank $(\partial \widetilde{x}^i / \partial x^j) = n$. The transformation rules (1) rewrite explicitly as

$$\begin{aligned}
&\widetilde{x}^1 = \widetilde{x}^1(x^1, ..., x^n),\ \widetilde{x}^2 = \widetilde{x}^2(x^1, ..., x^n), ...,\ \widetilde{x}^n = \widetilde{x}^n(x^1, ..., x^n),\\
&\widetilde{y}^1 = \partial_1^1 y^1 + \partial_2^1 y^2 + \partial_3^1 y^3 + ... + \partial_n^1 y^n\\
&\widetilde{y}^2 = \partial_1^2 y^1 + \partial_2^2 y^2 + \partial_3^2 y^3 + ... + \partial_n^2 y^n\\
&\vdots\\
&\widetilde{y}^n = \partial_1^n y^1 + \partial_2^n y^2 + \partial_3^n y^3 + ... + \partial_n^n y^n,
\end{aligned} \tag{2}$$

where we used the notations $\partial_j^i := \partial \widetilde{x}^i / \partial x^j$, and we have det $\left(\partial_j^i\right) \neq 0$.

Let us consider now the Berwald-Moór metric of order n on the tangent bundle TM^n, which is expressed by

$$F_n(y) = \sqrt[n]{y^1 y^2 y^3 ... y^n}. \tag{3}$$

To have a global geometrical character of the Berwald-Moór metric (3), we must have $F_n(\widetilde{y}) = F_n(y)$. It means that $\widetilde{y}^1 \widetilde{y}^2 ... \widetilde{y}^n = y^1 y^2 ... y^n$.

Theorem 1 *For $n \geq 3$, a transformation of coordinates (2) invariates the Berwald-Moór metric (3) if and only if there exist some arbitrary real numbers $a_1, a_2, ..., a_n$ verifying the equality $\Pi_{i=1}^n a_i = 1$, together with a permutation $\sigma \in S_n$ of the set $\{1, 2, ..., n\}$, such that*

$$\widetilde{\mathcal{X}} = \mathcal{P}_\sigma \left(a_1, a_2, ..., a_n\right) \cdot \mathcal{X} + \mathcal{X}_0, \tag{4}$$

where

$$\mathcal{X}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \\ \vdots \\ x_0^n \end{pmatrix} \in M_{n,1}(\mathbb{R}), \qquad \mathcal{X} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}, \qquad \widetilde{\mathcal{X}} = \begin{pmatrix} \widetilde{x}^1 \\ \widetilde{x}^2 \\ \vdots \\ \widetilde{x}^n \end{pmatrix},$$

and the matrix $\mathcal{P}_\sigma \left(a_1, a_2, ..., a_n\right)$ has all entries equal to zero except the entries

$$p_{1\sigma(1)} = a_1,\ p_{2\sigma(2)} = a_2,\ \dots\ ,\ p_{n\sigma(n)} = a_n.$$

Proof. The transformation of coordinates (2) invariates the Berwald-Moór metric (3) if and only if

$$\begin{cases} \sum_{\tau \in S_n} \partial_{\tau(1)}^1 \partial_{\tau(2)}^2 ... \partial_{\tau(n)}^n = 1 \\ \partial_{k_1}^1 \partial_{k_2}^2 ... \partial_{k_n}^n = 0, \end{cases} \tag{5}$$

for any $k_1, k_2, ..., k_n \in \{1, 2, ..., n\}$ such that $\{k_1, k_2, ..., k_n\} \neq \{1, 2, ..., n\}$ (this means that at least two indices k_r and k_s are equal). Then, there exists a permutation σ of the set $\{1, 2, ..., n\}$ such that

$$\partial^i_{\sigma(i)} \neq 0, \quad \forall\ i = \overline{1, n}.$$

This is because if we suppose that there exists an index $i_0 \in \{1, 2, ..., n\}$ such that $\partial^{i_0}_{\sigma(i_0)} = 0$ for any permutation $\sigma \in S_n$, then (using the first equation of the system (5)) we deduce that we have $0 = 1$. Contradiction!

Let us prove now that for any $i \in \{1, 2, ..., n\}$ we have

$$\partial^i_k = 0, \quad \forall\ k \in \{1, 2, ..., n\} \backslash \{\sigma(i)\}.$$

On the contrary, suppose there exist $i, k \in \{1, 2, ..., n\}$, with $k \neq \sigma(i)$, such that $\partial^i_k \neq 0$. Because we obviously have $k = \sigma(j)$, where $j \neq i$, it follows that there exist two different indices $i, j \in \{1, 2, ..., n\}$ such that $\partial^i_{\sigma(j)} \neq 0$. Because we have the inequality $n \geq 3$, it follows that there exists an arbitrary third index $q \in \{1, 2, ..., n\}$ which is different by i and j. Consequently, applying the second equation from the system (5) for an arbitrary index $k_q \in \{1, 2, ..., n\}$ and for

$$k_i = k_j = \sigma(j), \quad k_p = \sigma(p),\ \forall\ p \in \{1, 2, ..., n\} \backslash \{i, j, q\},$$

we find

$$\underbrace{\partial^1_{\sigma(1)} ... \partial^{i-1}_{\sigma(i-1)} \partial^i_{\sigma(j)} \partial^{i+1}_{\sigma(i+1)} ... \partial^{j-1}_{\sigma(j-1)} \partial^j_{\sigma(j)} \partial^{j+1}_{\sigma(j+1)} ... \partial^{q-1}_{\sigma(q-1)}}_{\nparallel \atop 0} \partial^q_{k_q} \underbrace{\partial^{q+1}_{\sigma(q+1)} ... \partial^n_{\sigma(n)}}_{\nparallel \atop 0} = 0.$$

It follows that we have

$$\partial^q_{k_q} = 0, \quad \forall\ k_q = \overline{1, n}.$$

This implies that $\det\left(\partial^i_j\right) = 0$. Contradiction!

In conclusion, we deduce that a transformation of coordinates (2) invariates the Berwald-Moór metric (3) if and only if it satisfies the following conditions:

$$\partial^1_{\sigma(1)} \partial^2_{\sigma(2)} ... \partial^n_{\sigma(n)} = 1, \tag{6}$$

$$\partial^i_1 = \partial^i_2 = ... = \partial^i_{\sigma(i)-1} = \partial^i_{\sigma(i)+1} = ... = \partial^i_{n-1} = \partial^i_n = 0, \quad \forall\ i = \overline{1, n}. \tag{7}$$

The equations (6) and (7) imply that $\tilde{x}^i = \tilde{x}^i(x^{\sigma(i)})$ and

$$\partial^i_{\sigma(i)} = p_{i\sigma(i)} := a_i \in \mathbb{R} \backslash \{0\}.$$

In other words, we get the affine transformations (no sum by i):

$$\tilde{x}^i = p_{i\sigma(i)} x^{\sigma(i)} + x^i_0, \qquad \forall\ i = \overline{1, n},$$

where $x^i_0 \in \mathbb{R}$, $\forall\ i = \overline{1, n}$. ■

Corollary 2 *The set of the local transformations of coordinates that invariates the Berwald-Moór metric (3) has an algebraic structure of a non-abelian group with respect to the operation of composition of functions.*

Proof. Firstly, it is important to notice that the following matrix properties are true:

$$\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) = \left(a_i \delta_{j\sigma(i)}\right)_{i,j=\overline{1,n}},$$

$$\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) = \mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) \Leftrightarrow \sigma = \tau \text{ and } a_i = b_i \ \forall\ i = \overline{1,n},$$

$$\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) = \mathcal{P}_{\tau\circ\sigma}\left(a_1 b_{\sigma(1)}, a_2 b_{\sigma(2)}, ..., a_n b_{\sigma(n)}\right),$$

$$\mathcal{P}_e\left(1, 1, ..., 1\right) = I_n, \qquad \det\left[\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right)\right] = \varepsilon(\sigma),$$

$$\left[\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right)\right]^{-1} = \mathcal{P}_{\sigma^{-1}}\left(a^{-1}_{\sigma^{-1}(1)}, a^{-1}_{\sigma^{-1}(2)}, ..., a^{-1}_{\sigma^{-1}(n)}\right),$$

where e is the identity permutation, I_n is the identity matrix, and $\varepsilon(\sigma)$ is the signature of the permutation σ.

Let $\mathfrak{T}$ be the set of the local transformations of coordinates that invariates the Berwald-Moór metric (3). Let also $\mathcal{S}$ and $\mathcal{T}$ be two arbitrary transformations from $\mathfrak{T}$. Then we have

$$\mathcal{S} : \tilde{\mathcal{X}} = \mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \mathcal{X} + \mathcal{X}_0,$$
$$\mathcal{T} : \mathcal{X} = \mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) \cdot \overline{\mathcal{X}} + \overline{\mathcal{X}}_0$$

and

$$\mathcal{S}\circ\mathcal{T} : \widetilde{\mathcal{X}} = \mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \left[\mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) \cdot \overline{\mathcal{X}} + \overline{\mathcal{X}}_0\right] + \mathcal{X}_0 =$$

$$= \left[\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right)\right] \cdot \overline{\mathcal{X}} + \overline{\mathcal{X}}_1 =$$
$$= \mathcal{P}_{\tau\circ\sigma}\left(a_1 b_{\sigma(1)}, a_2 b_{\sigma(2)}, ..., a_n b_{\sigma(n)}\right) \cdot \overline{\mathcal{X}} + \overline{\mathcal{X}}_1,$$

where

$$\overline{\mathcal{X}}_1 = \mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \overline{\mathcal{X}}_0 + \mathcal{X}_0.$$

Moreover, the neutral transformation element is

$$\mathcal{E} : \tilde{\mathcal{X}} = \mathcal{P}_e\left(1, 1, ..., 1\right) \cdot \mathcal{X},$$

and we have

$$\mathcal{S}^{-1} : \mathcal{X} = \left[\mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right)\right]^{-1} \cdot \left[\widetilde{\mathcal{X}} - \mathcal{X}_0\right] =$$

$$= \mathcal{P}_{\sigma^{-1}}\left(a^{-1}_{\sigma^{-1}(1)}, a^{-1}_{\sigma^{-1}(2)}, ..., a^{-1}_{\sigma^{-1}(n)}\right) \cdot \widetilde{\mathcal{X}} + \widetilde{\mathcal{X}}_1,$$

where

$$\widetilde{\mathcal{X}}_1 = -\mathcal{P}_{\sigma^{-1}}\left(a^{-1}_{\sigma^{-1}(1)}, a^{-1}_{\sigma^{-1}(2)}, ..., a^{-1}_{\sigma^{-1}(n)}\right) \cdot \mathcal{X}_0.$$

In conclusion, the set $\mathfrak{T}$ is a group with respect to the operation of composition of functions.

Let σ and τ be two permutations such that $\tau\circ\sigma \neq \sigma\circ\tau$. If we take, for instance, $\mathcal{S}_0$ and $\mathcal{T}_0$ two transformations from $\mathfrak{T}$ having the homogenous form

$$\mathcal{S}_0(\mathcal{X}) = \mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \mathcal{X}, \qquad \mathcal{T}_0(\mathcal{X}) = \mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) \cdot \mathcal{X},$$

then we get

$$\left(\mathcal{S}_0\circ\mathcal{T}_0\right)(\mathcal{X}) = \mathcal{P}_\sigma\left(a_1, a_2, ..., a_n\right) \cdot \left[\mathcal{P}_\tau\left(b_1, b_2, ..., b_n\right) \cdot \mathcal{X}\right] =$$

$$= [\mathcal{P}_\sigma(a_1, a_2, ..., a_n) \cdot \mathcal{P}_\tau(b_1, b_2, ..., b_n)] \cdot \mathcal{X} =$$
$$= \mathcal{P}_{\tau\circ\sigma}\left(a_1 b_{\sigma(1)}, a_2 b_{\sigma(2)}, ..., a_n b_{\sigma(n)}\right) \cdot \mathcal{X} \neq$$
$$\neq \mathcal{P}_{\sigma\circ\tau}\left(b_1 a_{\tau(1)}, b_2 a_{\tau(2)}, ..., b_n a_{\tau(n)}\right) \cdot \mathcal{X} =$$
$$= [\mathcal{P}_\tau(b_1, b_2, ..., b_n) \cdot \mathcal{P}_\sigma(a_1, a_2, ..., a_n)] \cdot \mathcal{X} = (\mathcal{T}_0 \circ \mathcal{S}_0)(\mathcal{X}).$$

Therefore, the group $(\mathfrak{T}, \circ)$ is non-abelian. ■

For any permutation $\sigma \in S_n$, let us denote by $\mathcal{W}_\sigma$ the set of all matrices of type $\mathcal{P}_\sigma(a_1, a_2, ..., a_n)$. We recall that such a matrix has all entries equal to zero except the entries $a_{1\sigma(1)} = a_1$, $a_{2\sigma(2)} = a_2$, ..., $a_{n\sigma(n)} = a_n$, verifying the equality $\Pi_{i=1}^n a_i = 1$. Let us consider the matrix

$$E_\sigma := \mathcal{P}_\sigma(1, 1, ..., 1) \in \mathcal{W}_\sigma.$$

In such a context, we can prove the following important algebraic result of characterization:

Proposition 3 *Let $\sigma \in S_n$ be an arbitrary permutation of the set $\{1, 2, ..., n\}$. Then, an arbitrary matrix $\mathfrak{X} \in M_n(\mathbb{R})$ belongs to the set $\mathcal{W}_{\sigma^{-1}}$ if and only if the following statements are true:*

(1) $\det(\mathfrak{X} \cdot E_\sigma) = 1$;

(2) *The vectors*

$$\mathbf{e_1} = (1, 0, 0, ..., 0),\ \mathbf{e_2} = (0, 1, 0, ..., 0),\ ...,\ \mathbf{e_n} = (0, 0, 0, ..., 1)$$

are eigenvectors of the matrix $(\mathfrak{X} \cdot E_\sigma)$.

Proof. An arbitrary matrix $A \in M_n(\mathbb{R})$ is a diagonal matrix of the form

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

with $\lambda_1 \lambda_2 ... \lambda_n = 1$, if and only if $\det A = 1$ and it has the vectors $\mathbf{e_1}$, $\mathbf{e_2}$, ..., $\mathbf{e_n}$ as eigenvectors. Using now the properties (1) and (2), we deduce that

$$\mathfrak{X} \cdot E_\sigma = A \Leftrightarrow \mathfrak{X} \cdot \mathcal{P}_\sigma(1, 1, ..., 1) = \mathcal{P}_e(\lambda_1, ..., \lambda_n) \Leftrightarrow$$
$$\Leftrightarrow \mathfrak{X} = \mathcal{P}_e(\lambda_1, ..., \lambda_n) \cdot [\mathcal{P}_\sigma(1, 1, ..., 1)]^{-1} =$$
$$= \mathcal{P}_e(\lambda_1, ..., \lambda_n) \cdot \mathcal{P}_{\sigma^{-1}}(1, 1, ..., 1) = \mathcal{P}_{\sigma^{-1}}(\lambda_1, ..., \lambda_n) \in \mathcal{W}_{\sigma^{-1}}.$$

■

In what follows we prove that the set of matrices $D_n^1(\mathbb{R}) := \mathcal{W}_e$, where e is the identity permutation, is a Lie group of dimension $(n-1)$ with respect to the multiplication of matrices.

Proposition 4 *The set of matrices $D_n^1(\mathbb{R})$ has a structure of commutative Lie group with respect to the multiplication of matrices. The dimension as manifold of the Lie group $D_n^1(\mathbb{R})$ is $(n-1)$, and the corresponding Lie algebra is*

$$L(D_n^1(\mathbb{R})) = \{A = \operatorname{diag}(a_1, ..., a_n) \mid \operatorname{Trace}(A) = 0\} := d_n^1(\mathbb{R}). \tag{8}$$

Proof. It is obvious that

$$D_n^1(\mathbb{R}) = \mathcal{W}_e = \{\mathcal{P}_e(a_1, a_2, ..., a_n) \mid a_i \in \mathbb{R}\} =$$
$$= \{A = \operatorname{diag}(a_1, ..., a_n) \mid \det A = a_1 \cdot ... \cdot a_n = 1\}$$

is a commutative subgroup of the special linear group $SL_n(\mathbb{R})$. It is also easy to see that we have

$$D_n^1(\mathbb{R}) = \left\{ A = \operatorname{diag}\left(a_1, ..., a_{n-1}, \frac{1}{a_1 \cdot ... \cdot a_{n-1}}\right) \;\middle|\; a_1, \ ..., \ a_{n-1} \in \mathbb{R}\backslash\{0\}\right\}.$$

Let $\phi_U : D_n^1(\mathbb{R}) \to U \subset M := \mathbb{R}^{n-1}\backslash \{(a_1, ..., a_{n-1}) \mid a_1 \cdot ... \cdot a_{n-1} = 0\}$ be the bijection defined by $\phi_U(A) = (a_1, ..., a_{n-1})$, where U is an arbitrary local chart on M. It folows that we can endow $D_n^1(\mathbb{R})$ with a differentiable structure of dimension $(n-1)$ such that all maps ϕ_U to become diffeomorfisms. Therefore, the mapping $\mu : D_n^1(\mathbb{R}) \times D_n^1(\mathbb{R}) \to D_n^1(\mathbb{R})$, defined by

$$\mu(A, B) = A^{-1} \cdot B,$$

can be locally rewritten as

$$\widetilde{\mu} : \mathbb{R}^{2n-2}\backslash \{(a_1, ..., a_{n-1}, b_1, ..., b_{n-1}) \mid a_1 \cdot ... \cdot a_{n-1} \cdot b_1 \cdot ... \cdot b_{n-1} = 0\} \to \mathbb{R}^{n-1},$$

where

$$\widetilde{\mu}(a_1, ..., a_{n-1}, b_1, ..., b_{n-1}) = \left(\frac{b_1}{a_1}, \frac{b_2}{a_2}, ..., \frac{b_{n-1}}{a_{n-1}}\right).$$

It is obvious that $\widetilde{\mu}$ is a smooth map on the open domain

$$D = \mathbb{R}^{2n-2}\backslash \{(a_1, ..., a_{n-1}, b_1, ..., b_{n-1}) \mid a_1 \cdot ... \cdot a_{n-1} \cdot b_1 \cdot ... \cdot b_{n-1} = 0\}.$$

In conclusion, $D_n^1(\mathbb{R})$ is a commutative Lie group of dimension $(n-1)$.

In order to compute the Lie algebra $L(D_n^1(\mathbb{R})) = d_n^1(\mathbb{R})$, let us consider an arbitrary curve

$$\alpha : (-\epsilon, \epsilon) \to D_n^1(\mathbb{R}), \qquad \alpha(t) = A(t) = \operatorname{diag}(a_1(t), ..., a_n(t)) \in D_n^1(\mathbb{R}),$$

where $\alpha(0) = A(0) = I_n$. It follows that we have

$$\dot{\alpha}(0) = \left.\frac{dA}{dt}\right|_{t=0} = \operatorname{diag}(\dot{a}_1(0), ..., \dot{a}_n(0)).$$

Because we have $\det A(t) = 1$, we deduce that

$$0 = \left.\frac{d\,[\det A]}{dt}\right|_{t=0} = \det[\operatorname{diag}(\dot{a}_1(0), 1, ..., 1)] +$$
$$+ \det[\operatorname{diag}(1, \dot{a}_2(0), 1, ..., 1)] + \ ... \ + \det[\operatorname{diag}(1, 1, ..., 1, \dot{a}_n(0))] =$$
$$= \dot{a}_1(0) + \dot{a}_1(0) + ... + \dot{a}_n(0).$$

This means that the Lie algebra of $D_n^1(\mathbb{R})$ is given by (8). ■

Remark 5 *A basis of the Lie algebra* $d_n^1(\mathbb{R})$ *is given by* $(E_i)_{i=\overline{1,n-1}}$, *where*

$$E_i = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \end{pmatrix}.$$

Notice that in the matrix E_i *the number* 1 *appears on the position* (i, i). *In other words, we can briefly write*

$$E_i = (\delta_{ri}\delta_{si} - \delta_{rn}\delta_{sn})_{r,s=\overline{1,n}}, \quad \forall\ i = \overline{1, n-1}.$$

Consequently, it is easy to see now that ***all structure constants of the Lie algebra*** $d_n^1(R)$ ***are equal to zero****.*

Acknowledgements. The authors of this paper thank Professor M. Păun for the useful geometrical discussions on this research topic.

References

[1] G.S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht, 1984.

[2] V. Balan, M. Neagu, *Jet Single-Time Lagrange Geometry and Its Applications*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2011.

[3] L. Berwald, *Über Finslersche und Cartansche Geometrie II*, Compositio Math., vol. **7** (1940), 141-176.

[4] É. Cartan, *La théorie des groupes finis et continus et l'analysis situs*, Mémorial des Sciences Mathématiques, vol. **42** (1952), 1-61.

[5] G.I. Garas'ko, *Foundations of Finsler Geometry for Physicists* (in Russian), Tetru Eds, Moscow, 2009.

[6] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[7] M. Matsumoto, H. Shimada, *On Finsler spaces with* 1*-form metric. II. Berwald-Moór's metric* $L = \left(y^1y^2...y^n\right)^{1/n}$, Tensor N.S., vol **32** (1978), 275-278.

[8] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, Dordrecht, 1994.

[9] A. Moór, *Über die Dualität von Finslerschen und Cartanschen Räumen*, Acta Math., vol. **88** (1952), 347-370.

[10] M. Neagu, *The Finsler-like geometry of the (t, x)-conformal deformation of the jet Berwald-Moór metric*, arXiv:1202.3521 [math-ph].

[11] M. Neagu, H. Raeisi-Dehkordi, *On the invariance groups of the Berwald-Moór metric of order two and three*, Preprint, 16 pages, September (2012).

[12] D.G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics **1** (**1**), vol. **1** (2004), 5-18.

[13] P.K. Rashevsky, *Polymetric Geometry* (in Russian), In: *"Proc. Sem. on Vector and Tensor Analysis and Applications in Geometry, Mechanics and Physics"* (Ed. V.F. Kagan), vol. **5**, M-L, OGIZ, 1941.

[14] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.